Worked Solutions

Edexcel C4 Paper E

1. (a) when $y = 1, 4x^2 + 3 = 12$

$$x^2 = \frac{9}{4}$$

$$x = \pm\frac{3}{2} \qquad \textbf{(2)}$$

(b) differentiating, $8x + 6y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0 \qquad \dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{8x}{6y} = -\dfrac{4x}{3y}$

at $\left(\frac{3}{2}, 1\right)$ gradient $= -\dfrac{4 \times \frac{3}{2}}{3} = -2$

at $\left(-\frac{3}{2}, 1\right)$ gradient $= 2$ **(4)**

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2. $(8+x)^{\frac{1}{3}} = \left[8\left(1+\frac{x}{8}\right)\right]^{\frac{1}{3}} = 2\left(1+\frac{x}{8}\right)^{\frac{1}{3}} = 2\left[1 + \frac{1}{3}\left(\frac{x}{8}\right) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2}\left(\frac{x}{8}\right)^2 + \dots\right]$

$$= 2 + \frac{x}{12} - \frac{1}{288}x^2 + \dots \qquad \textbf{(4)}$$

(b) for $\left(8 + 3m + m^2\right)^{\frac{1}{3}}$, let $3m + m^2 = x$

$$\left(8 + 3m + m^2\right)^{\frac{1}{3}} = 2 + \left(\frac{3m + m^2}{12}\right) - \frac{1}{288}\left(3m + m^2\right)^2$$

$$= 2 + \frac{1}{4}m + \frac{1}{12}m^2 - \frac{1}{288}\cdot 9m^2 + \dots$$

$$= 2 + \frac{1}{4}m + \frac{1}{12}m^2 - \frac{1}{32}m^2 = 2 + \frac{1}{4}m + \frac{5}{96}m^2. \qquad \textbf{(3)}$$

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3. (a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\frac{1}{2}\cdot 2\cos 2\theta}{-\sin\theta} = -\dfrac{\cos 2\theta}{\sin\theta}$

at $\theta = \dfrac{\pi}{6}$, gradient $= -\dfrac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{6}} = -\dfrac{\frac{1}{2}}{\frac{1}{2}} = -1$ **(2)**

(b) at $\theta = \dfrac{\pi}{6}$, $x = \cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$ and $y = \dfrac{1}{2}\sin\dfrac{\pi}{3} = \dfrac{1}{4}\sqrt{3}$.

equation of tangent is $y - \dfrac{\sqrt{3}}{4} = -1\left(x - \dfrac{\sqrt{3}}{2}\right)$

$$4y - \sqrt{3} = -4x + 2\sqrt{3}$$

$$4y + 4y = 3\sqrt{3} \qquad \textbf{(3)}$$

(c) $y^2 = \dfrac{1}{4}\sin^2 2\theta = \dfrac{1}{4}(2\sin\theta\cos\theta)^2 = \dfrac{1}{4}\cdot 4\sin^2\theta\cos^2\theta = (1-\cos^2\theta)\cos^2\theta$.

$\therefore \quad y^2 = (1-x^2)(x^2)$ **(3)**

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4. (a) $(0, 10)$ **(1)**

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -10(-k)\mathrm{e}^{-kx}$

at $x = 0$, gradient $= 10k$

$$10k = 5 \Rightarrow k = \frac{1}{2} \qquad \textbf{(3)}$$

(c) area $= \displaystyle\int_0^4 \left(20 - 10\,\mathrm{e}^{-\frac{1}{2}x}\right)\,\mathrm{d}x = \left[20x + 20\,\mathrm{e}^{-\frac{1}{2}x}\right]_0^4$

$$= 80 + 20\,\mathrm{e}^{-2} - (0 + 20) = 60 + \frac{20}{\mathrm{e}^2} \qquad \textbf{(5)}$$

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5. (a) when $t = 0, \theta = 300 - 270e^{\circ}$

$$\theta = 30 \qquad \textbf{(1)}$$

(b) as $t \to \infty, \theta \to 300$ **(2)**

(c) $200 = 300 - 270\,e^{-0.05t}$

$$270\,e^{-0.05t} = 100$$

$$\ln e^{-0.05t} = \ln\left(\frac{100}{270}\right)$$

$$-0.05t = \ln\left(\frac{100}{270}\right)$$

$$t = 19.9 \text{ minutes} \qquad \textbf{(3)}$$

(d) $\dfrac{d\theta}{dt} = -270(-0.05)\,e^{-0.05t}$

when $t = 2$, $\dfrac{d\theta}{dt} = 270 \times 0.05 \times e^{-0.1}$

$$= 12.2^{\circ}\text{ C/min} \qquad \textbf{(3)}$$

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6. (a) $\dfrac{2}{1-x} - \dfrac{2}{2-x}$ (‘cover up’ rule) **(2)**

(b) we have $\dfrac{2}{1-x} - \dfrac{2}{2(1-\frac{x}{2})} = 2(1-x)^{-1} - \left(1 - \dfrac{x}{2}\right)^{-1}$

$$= 2\left[1 + (-1)(-x) + \frac{(-1)(-2)}{2}(-x)^2 + \ldots\right]$$

$$-\left[1 + (-1)\left(\frac{-x}{2}\right) + \frac{(-1)(-2)}{2}\left(\frac{-x}{2}\right)^2 + \ldots\right]$$

$$= 2 + 2x + 2x^2 - 1 - \frac{x}{2} - \frac{x^2}{4}$$

$$= 1 + \frac{3}{2}x + \frac{7}{4}x^2 \qquad \textbf{(6)}$$

(c) $\displaystyle\int_0^{\frac{1}{2}} \left(\frac{2}{1-x} - \frac{2}{2-x}\right) dx = \Big[-2\ln(1-x) + 2\ln(2-x)\Big]_0^{\frac{1}{2}}$

$$= -2\ln\frac{1}{2} + 2\ln\frac{3}{2} - (-2\ln 1 + 2\ln 2)$$

$$= -2\ln 2^{-1} + 2\ln\frac{3}{2} - 0 - 2\ln 2 = 2\ln\frac{3}{2} \qquad \textbf{(5)}$$

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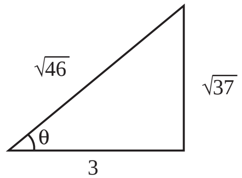
7. (a) let $\angle ABC = \theta$, $\overrightarrow{BA} = \begin{pmatrix} -4 \\ 0 \\ -3 \end{pmatrix}$, $|\overrightarrow{BA}| = \sqrt{4^2 + 3^2} = 5$

$$\overrightarrow{BC} = \begin{pmatrix} -6 \\ 1 \\ 3 \end{pmatrix}, \; |\overrightarrow{BC}| = \sqrt{6^2 + 1^2 + 3^2} = \sqrt{46}$$

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = |\overrightarrow{BA}| \times |\overrightarrow{BC}| \cos\theta$$

$$\begin{pmatrix} -4 \\ 0 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -6 \\ 1 \\ 3 \end{pmatrix} = 5\sqrt{46}\cos\theta \qquad 24 - 9 = 5\sqrt{46}\cos\theta \qquad \cos\theta = \frac{3}{\sqrt{46}} \qquad \textbf{(4)}$$

(b) area of $\triangle ABC = \dfrac{1}{2}|\overrightarrow{BA}| \times |\overrightarrow{BC}| \sin\theta$



$$\text{area} = \frac{1}{2} \cdot 5 \times \sqrt{46} \cdot \frac{\sqrt{37}}{\sqrt{46}}$$

$$= \frac{5}{2}\sqrt{37} \qquad \sin\theta = \frac{\sqrt{37}}{\sqrt{46}}. \qquad \textbf{(4)}$$

(c) $\overrightarrow{AC} = \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}$ $\quad \overrightarrow{OD} = \begin{pmatrix} 4 \\ -2 \\ -12 \end{pmatrix}$

$\overrightarrow{OD} = -2\begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}$, so AC is parallel to OD. **(2)**

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8. (*a*) Let $I = \int_0^1 \frac{x}{(2x+1)^2}\,\mathrm{d}x$

let $t = 2x + 1$

$\frac{\mathrm{d}t}{\mathrm{d}x} = 2 \Rightarrow \mathrm{d}x = \frac{1}{2}\mathrm{d}t$

$x = \frac{1}{2}(t-1)$

when $x = 1, \quad t = 3$
$x = 0, \quad t = 1$

$$\therefore \quad I = \frac{1}{2}\int_1^3 \frac{1}{2}\frac{(t-1)}{t^2}\mathrm{d}t$$

$$= \frac{1}{4}\int_1^3 \left(\frac{1}{t} - t^{-2}\right)\mathrm{d}t = \frac{1}{4}\left[\ln t + \frac{1}{t}\right]_1^3$$

$$= \frac{1}{4}\left[\ln 3 + \frac{1}{3} - (\ln 1 + 1)\right]$$

$$= \frac{1}{4}\left[\ln 3 - \frac{2}{3}\right]$$

(7)

(*b*) $$\int_1^{\mathrm{e}} x^2 \ln x \;\mathrm{d}x = \int_1^{\mathrm{e}} \ln x \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^3}{3}\right)\mathrm{d}x$$

$$= \left[\frac{x^3}{3}\ln x\right]_1^{\mathrm{e}} - \int_1^{\mathrm{e}} \frac{x^3}{3}\cdot\frac{1}{x}\;\mathrm{d}x$$

$$= \left(\frac{\mathrm{e}^3}{3}\ln \mathrm{e} - 0\right) - \left[\frac{x^3}{9}\right]_1^{\mathrm{e}}$$

$$= \frac{\mathrm{e}^3}{3} - \left(\frac{\mathrm{e}^3}{9} - \frac{1}{9}\right) = \frac{3\mathrm{e}^3 - \mathrm{e}^3 + 1}{9}$$

$$= \frac{2\mathrm{e}^3 + 1}{9}$$

(6)